Investigation of a Terminal Guidance System for a Satellite Rendezvous

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A guidance scheme for the terminal phase of a satellite rendezvous operation is presented. The terminal phase is that part of the rendezvous operation during which the final velocity and position corrections are made, exclusive of the actual contact maneuvers. The initial relative position and velocity between the space vehicle and the satellite are assumed to result from the normal dispersion during powered phase and orbit injection of an inertially guided ballistic rocket launched from the earth. The analysis is based on a nominal maximum initial separation of 5000 m and a maximum relative velocity of 50 m/sec in an arbitrary direction. Thrust acceleration levels well above the tidal acceleration level are assumed. This permits the problem to be reduced to a free space case, with tidal acceleration entering as a relatively small perturbation. A relative coordinate system is used in the analysis. Guidance equations for an exponential type of homing are developed. This, of course, implies that variable thrust rocket motors are employed. The response, propellant consumption, and effects of tidal acceleration are presented in graphical form.

Nomenclature

a = damping coefficient, $\sec^{-1}$
a_ξ = ξ component of tidal acceleration, m/sec^2
a_η = η component of tidal acceleration, m/sec^2
b = restoring force control coefficient, $\sec^{-2}$
F = total thrust on vehicle, kg
F_x = x component of thrust, kg
F_y = y component of thrust, kg
F_r = radial component of thrust, kg
F_ϕ = angular component of thrust, kg
g = standard acceleration of gravity, 9.81, m/sec^2
I_{sp} = specific impulse of thrust units, sec
k = parameter $\alpha x_0/V_0$
n = relative thrust level b/p
p = tidal acceleration coefficient $2gR_e^2/R_0^3$, $\sec^{-2}$
r = radial distance between satellite and vehicle, m
R = distance of vehicle from center of earth, m
R_e = radius of earth, 6.377×10^6, m
R_0 = radius of satellite orbit, 6.946×10^6, m
t = time from ignition of terminal phase, sec
t_1 = value of t when integrand in propellant consumption integrals is zero, sec
u_0 = initial velocity in x direction ($\dot{x}_0$), m/sec
V = magnitude of velocity relative to satellite, m/sec
V_0 = magnitude of initial velocity relative to satellite, m/sec
v_0 = initial velocity in y direction ($\dot{y}_0$), m/sec
W_p = weight of propellant consumed, kg
W = weight of vehicle (constant), kg
x, y = space orientation fixed rectangular coordinates of vehicle relative to satellite, m
x_0, y_0 = initial position of vehicle relative to satellite, m
x_c = position of vehicle in x direction, relative to satellite, at injection phase cutoff, m
α = decay constant, or another form of damping control coefficient; $\alpha = a/2$, $\sec^{-1}$
β = argument coefficient in hyperbolic functions, $\sec^{-1}$
β_1 = $(p/2)^{1/2}$, $\sec^{-1}$
γ = angle between line of sight and thrust vector of vehicle in polar coordinates about satellite, deg
ξ, η = space orientation fixed rectangular coordinates of vehicle relative to satellite, with η axis coincident with earth's radius vector at time of terminal phase ignition, m
θ = initial angle between velocity and radius vectors of vehicle, in x-y or r-ϕ coordinates, deg
θ_c = same angle as θ, but at injection phase cutoff, deg
σ = damping factor
σ_ξ, σ_η = damping factor as identified by subscript
ϕ = angular coordinate in polar coordinate representation, rad
ω = angular velocity of vehicle motion around satellite; $\omega = \dot{\phi}$, rad/sec

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Introduction

As the technology of orbiting satellites develops, the possibility of rendezvous between a satellite and a space vehicle becomes more imminent. Such a rendezvous may be required to refuel, capture, and adjust the orbit of, supplement instrumentation or structure of, or disrupt the operation of a satellite. Manned satellites and space vehicles will rendezvous to transfer passengers, supplies, equipment, etc.

In this analysis, those terminal maneuvers are investigated which occur after the space vehicle is injected into an orbit near the satellite. The initial relative position and velocity between the satellite and space vehicle are those which may occur when an inertially guided ballistic rocket is used to place a space vehicle in rendezvous with a satellite whose orbit is known from tracking data. Under these conditions, small displacement and velocity errors arise from the boost phase guidance system, atmospheric deviations, and uncertainty of the tracking data.

The space vehicle is assumed to contain equipment that locates the satellite at initiation of the terminal phase and provides the guidance information. The guidance information consists of 1) separation distance, 2) its time rate of change, and 3) angular velocity of the line of sight with respect to inertial space. A radar or infrared system may be used to provide the distance and rate of separation. The rate of rotation of the line of sight may be referenced from a gyro or the stars. Guidance computations may be performed on the vehicle, satellite, or the earth.

The guidance problem is first analyzed by considering the components of motion in two mutually perpendicular directions, neglecting tidal acceleration (rendezvous in free space). Then the equations of motion are transformed to polar coordinates, the form in which the guidance information appears. The analysis then continues with the determination of propellant consumption in free space, consideration of delayed ignition, and effects of tidal acceleration.

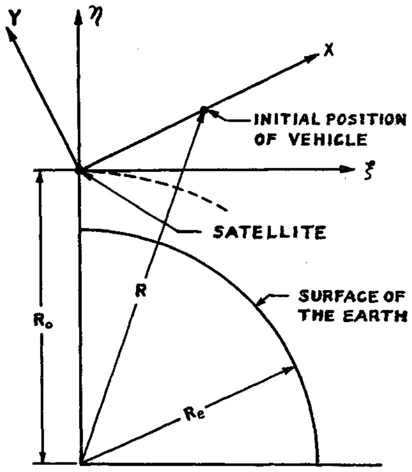


Fig. 1 Space direction fixed coordinates with origin at satellite; direction of η axis is along earth's radius vector at ignition

The numerical results and graphs are based on a 96-min circular satellite orbit. Maximum initial separation and velocity of the vehicle relative to the satellite are taken as 5000 m and 50 m/sec, respectively. A specific impulse of 255 sec is assumed for the propellant.

Discussions of several alternate terminal guidance schemes for satellite rendezvous may be found in Refs. 1–4. From these articles, background information pertaining to tracking methods, configurations, system requirements, computers, sensing schemes, etc., may be obtained.

Guidance Concept

Basic Assumptions

An exponential type of guidance, with damping factor of unity or greater, is assumed. This type of guidance lends itself to easy analysis, which, even though not strictly employed in the actual vehicle, will give a good representation of the requirements for terminal guidance.

Chemically fueled variable thrust rocket motors are assumed for the propulsive system of the rendezvous vehicle. The response lag in thrust orientation is neglected. The mass of the rendezvous vehicle is assumed to be constant, since the propellant consumption is low in comparison with the vehicle weight.

Rectangular Coordinates

The guidance system is equivalent to one-dimensional exponential guidance in each of two mutually perpendicular directions, x and y in a relative coordinate system. The relative coordinate system is defined by 1) satellite at the origin at all times; 2) x axis directed from satellite through initial position of space vehicle; 3) y axis at 90° to x axis, lying in plane containing x axis and relative velocity vector; and 4) the x-y axes are space orientation fixed.

The coordinate system is shown in Fig. 1. Tidal acceleration is neglected in these equations. The various equations in the two directions are:

Guidance Equations

$$F_x = (W/g)(a\dot{x} + bx) \qquad F_y = (W/g)(a\dot{y} + by) \tag{1}$$

Equations of Motion

$$\ddot{x} + a\dot{x} + bx = 0 \qquad \ddot{y} + a\dot{y} + by = 0 \tag{2}$$

Initial Conditions

$$x = x_0 \qquad \dot{x} = u_0 = V_0 \cos\theta \qquad y = y_0 \qquad \dot{y} = v_0 = V_0 \sin\theta \tag{3}$$

where

F_x = thrust component in x direction
F_y = thrust component in y direction
W = weight of the vehicle
a, b = guidance coefficients
V_0 = magnitude of the initial velocity
θ = direction of V_0, from positive x axis toward positive y axis

The solutions of Eqs. (2) and their derivatives, with initial conditions (3), are as follows:

$$x = x_0 e^{-\alpha t}\left[\cosh\beta t + \frac{\alpha}{\beta}\left(1 + \frac{\cos\theta}{k}\right)\sinh\beta t\right]$$

$$\dot{x} = \alpha x_0 e^{-\alpha t}\left[\frac{\cos\theta}{k}\cosh\beta t - \frac{\alpha}{\beta}\left(1 - \frac{\beta^2}{\alpha^2} + \frac{\cos\theta}{k}\right)\sinh\beta t\right] \tag{4}$$

$$\ddot{x} = \alpha^2 x_0 e^{-\alpha t}\left[-\left(1 - \frac{\beta^2}{\alpha^2} + \frac{2\cos\theta}{K}\right)\cosh\beta t + \frac{1}{\alpha\beta}\left(\alpha^2 - \beta^2 + \frac{\alpha^2+\beta^2}{k}\cos\theta\right)\sinh\beta t\right]$$

$$y = (\alpha x_0/\beta k)\sin\theta\cdot e^{-\alpha t}\sinh\beta t$$

$$\dot{y} = (\alpha x_0/k)\sin\theta\cdot e^{-\alpha t}[\cosh\beta t - (\alpha/\beta)\sinh\beta t] \tag{5}$$

$$\ddot{y} = (\alpha x_0/k)\sin\theta\cdot e^{-\alpha t}\{-2\alpha\cosh\beta t + [(\alpha^2+\beta^2)/\beta]\sinh\beta t\}$$

where

$$\alpha = a/2 \qquad \beta = (\alpha^2 - b)^{1/2}$$
$$k = \alpha x_0/V_0 \qquad \text{damping factor } \sigma = \alpha/b^{1/2}$$

Overshoot occurs in x if $-(\cos\theta)/k > 1 + \beta/\alpha$. No overshoot occurs in the y direction.

At critical damping ($\sigma = 1$), Eqs. (4) and (5) reduce to

$$x = x_0 e^{-\alpha t}\left[1 + \left(1 + \frac{\cos\theta}{k}\right)\alpha t\right]$$

$$\dot{x} = \alpha x_0 e^{-\alpha t}\left[\frac{\cos\theta}{k} - \left(1 + \frac{\cos\theta}{k}\right)\alpha t\right] \tag{6}$$

$$\ddot{x} = -\alpha^2 x_0 e^{-\alpha t}\left[\left(1 + \frac{2\cos\theta}{k}\right) - \left(1 - \frac{\cos\theta}{k}\right)\alpha t\right]$$

$$y = x_0(\sin\theta/k)\cdot\alpha t e^{-\alpha t}$$

$$\dot{y} = \alpha x_0(\sin\theta/k)(1 - \alpha t)e^{-\alpha t} \tag{7}$$

$$\ddot{y} = -\alpha^2 x_0(\sin\theta/k)(2 - \alpha t)e^{-\alpha t}$$

$$\alpha = a/2 = b^{1/2} \qquad \beta = 0$$

Overshoot occurs in x if $-(\cos\theta)/k > 1$.

Paths of the vehicle for various values of k and θ are shown in Figs. 2 and 3. The damping factor in each case is unity. It is noted that the vehicle wanders farther out as the parameter k is decreased.

The points of constant αt form semicircles, as shown by the broken lines. This is shown easily as follows. From the first of Eqs. (6) and (7), one gets

$$[x - x_0(1 + \alpha t)e^{-\alpha t}]^2 + y^2 = [(x_0\alpha t/k)e^{-\alpha t}]^2 \tag{8}$$

which is an equation of a circle for each k and αt. The center is displaced to the right of the origin by the distance $x_0(1 + \alpha t)\exp(-\alpha t)$, the radius (squared) being given by the expression on the right-hand side of Eq. (8).

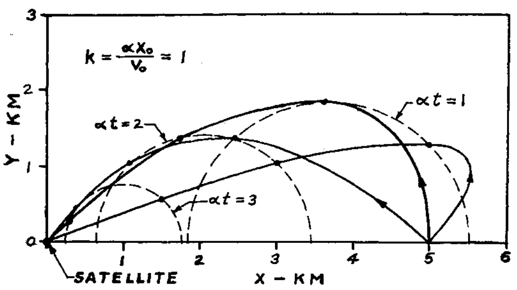


Fig. 2 Paths of vehicle for various initial directions of velocity vector; parameter $k = 1$

Theoretically, the time of rendezvous to the origin is infinite. Actually, this rendezvous phase terminates at some finite distance from the origin, say 50 m, and hence the rendezvous time is regarded as finite. The time of rendezvous tends to be long, since the velocities near the origin are small.

Polar Coordinates

The guidance information is sensed directly in polar coordinates as radial distance (between satellite and vehicle), radial veocity, and angular velocity of the radius vector with respect to inertial space. This information is sufficient for guidance purposes. The polar coordinate system is shown in Fig. 4. The vehicle will follow the paths described by Figs. 2 and 3, provided that a guidance scheme in polar coordinates is made equivalent to the rectangular coordinate scheme described by Eqs. (2–7).

Now, the guidance equations in polar coordinate form will be derived. Transformation of Eqs. (2) to polar coordinates, with $x = r\cos\phi$, $y = r\sin\phi$, and $\omega = \dot{\phi}$, yields

$$[\ddot{r} + a\dot{r} + (b - \omega^2)r]\cos\phi - [2\dot{r}\omega + (a\omega + \dot{\omega})r]\sin\phi = 0$$
$$[\ddot{r} + a\dot{r} + (b - \omega^2)r]\sin\phi + [2\dot{r}\omega + (a\omega + \dot{\omega})r]\cos\phi = 0 \quad (9)$$

from which one gets the equations of motion

$$\ddot{r} + ar + (b - \omega^2)r = 0$$
$$2\dot{r}\omega + (a\omega + \dot{\omega})r = 0 \quad (10)$$

The solutions of these equations are quite involved. There is no need for these solutions, however, since an ample description of the paths is given in rectangular coordinates.

Note that, if the guidance coefficients in the x and y directions differ, then the values of a and b in the first of Eqs. (9) differ from those in the second of Eqs. (9). In this case, the solution of these equations will contain ϕ.

The radial and angular components of the thrust vector (F_r and F_ϕ, respectively) easily are shown to be

$$F_r = (W/g)(\ddot{x}\cos\phi + \ddot{y}\sin\phi)$$
$$F_\phi = (W/g)(-\ddot{x}\sin\phi + \ddot{y}\cos\phi) \quad (11)$$

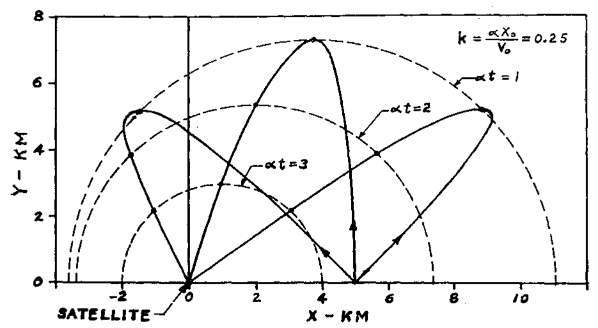


Fig. 3 Paths of vehicle for various initial directions of velocity vector; parameter $k = 0.25$

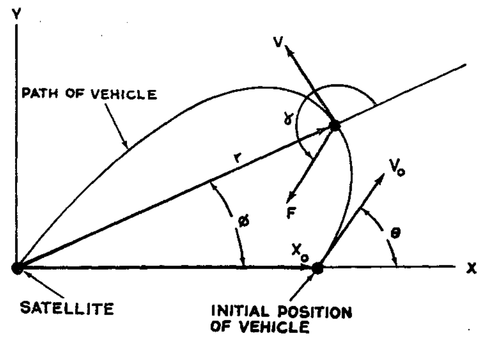


Fig. 4 Rectangular and polar coordinate in relative coordinate system; the x-y axes are space direction fixed

which in polar coordinates become

$$F_r = (W/g)(\ddot{r} - r\omega^2)$$
$$F_\phi = (W/g)(2\dot{r}\omega + r\dot{\omega}) \quad (12)$$

The thrust component now can be expressed in terms of the guidance coefficients by substituting Eqs. (10) into Eqs. (12). Hence, one has

$$F_r = -(W/g)(a\dot{r} + br)$$
$$F_\phi = -(W/g)ar\omega \quad (13)$$

Since the total thrust magnitude is $F = (F_r^2 + F_\phi^2)^{1/2}$, one immediately obtains:

Thrust Magnitude

$$F = (W/g)[(a\dot{r} + br)^2 + (ar\omega)^2]^{1/2} \quad (14a)$$

Direction of Thrust Vector

$$\gamma = \tan^{-1}\{[-ar\omega/-(a\dot{r} + br)]\} \quad (14b)$$

where γ is the angle between the thrust and radius vector (see Fig. 4).

Since ω is always positive and F_ϕ is always negative, the angle γ always must be between 180° and 360°. These are the guidance equations according to which the magnitude and direction of the thrust are controlled in terms of the sensed signals r, $\dot{r}$, and ω.

In the case of critical damping [$\sigma = 1$ and $a = 2(b)^{1/2}$], guidance Eqs. (14) reduce to:

Thrust Magnitude

$$F = (W/g)b^{1/2}[(2\dot{r} + b^{1/2}r)^2 + (2r\omega)^2]^{1/2} \quad (15a)$$

Direction of Thrust Vector

$$\gamma = \tan^{-1}[(-2r\omega)/-(2\dot{r} + b^{1/2}r)] \quad (15b)$$

In actual rendezvous, the perturbations due to the tidal acceleration cause the vehicle to deviate somewhat from the paths calculated for the unperturbed cases. The exact path is unimportant, as long as a soft rendezvous is accomplished. However, if the thrust level approaches the tidal acceleration level, the rendezvous conditions and propellant consumption may differ considerably from the unperturbed case. In this case, the tidal acceleration must be included in the equations of motion.

Propellant Consumption

Free Space Calculation

In the determination of propellant consumption, it will be assumed that a single thrust unit or several thrust units parallel to each other are employed. Though an actual system has an orientation response lag, instantaneous response is

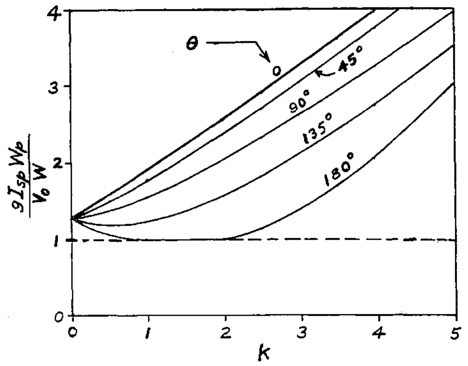


Fig. 5 Propellant consumption, in parametric form, as a function of the parameter $k = \alpha\, x_0/V_0$ for various directions of initial velocity vector; tidal acceleration is neglected

assumed. The thrust vector, therefore, will correspond to the commanded signal, and the rate of propellant consumption will be proportional to the total thrust. For convenience, the propellant consumption calculations will be made to the origin (infinite time), which is sufficiently accurate for the present purposes.

The equation for propellant consumption from ignition to interception, in kilograms, then will be

$$W_p = \frac{1}{I_{sp}} \int_0^\infty |F|dt = \frac{W}{gI_{sp}} \int_0^\infty (\ddot{x}^2 + \ddot{y}^2)^{1/2}dt \quad (16)$$

For a unit damping factor ($\sigma = 1$), the integral becomes

$$W_p = \frac{V_0 W}{gI_{sp}} \int_0^\infty e^{-\alpha t}\{[(k + 2\cos\theta) - (k + \cos\theta)\alpha t]^2 + (2 - \alpha t)^2 \sin^2\theta\}^{1/2} \cdot d(\alpha t) \quad (17)$$

In the cases when $\theta = 0$ and $180°$, the integrals are

$$\begin{aligned} gI_{sp}W/V_0W &= |\cos\theta + 2(k + \cos\theta)e^{-\alpha t_1}| \qquad \alpha t_1 > 0 \\ gI_{sp}W_p/(V_0W) &= 1 \qquad \alpha t_1 \leqslant 0 \end{aligned} \quad (18)$$

where

$$\alpha t_1 = (k + 2\cos\theta)/(k + \cos\theta)$$

Equations (18) give the propellant consumption for one-dimensional motion ($\theta = 0$ and $180°$) for $\sigma = 1$.

Curves of the propellant consumption parameter, $gI_{sp}W_p/(WV_0)$, as a function of θ and k are shown in Fig. 5. The horizontal line represents the basic propellant consumption, or the amount of propellant required to reduce the velocity from V_0 to zero in a linear direction.

The formula for the basic propellant consumption is

$$W_p \text{ (basic)} = V_0 W/(gI_{sp})$$

The propellant consumption in fractional form can be obtained by the following formula:

$$\frac{W_p}{W} = \frac{V_0}{gI_{sp}} \cdot \frac{gI_{sp}W_p}{WV_0} = \frac{\text{weight of propellant consumed}}{\text{total weight of vehicle}} \quad (19)$$

Figure 6 shows the propellant consumption directly as a percentage of the vehicle weight as a function of the initial velocity, for two values of the initial distance x_0. The relative thrust level n enters as a parameter. Curves for $\theta = 0$ and $180°$ are shown. The values for intermediate angles will lie between these two extremes.

These curves show that, for maximum initial conditions of 5000 m and 50 m/sec and a relative thrust level up to 16, the maximum propellant consumption is approximately 3% of the vehicle weight.

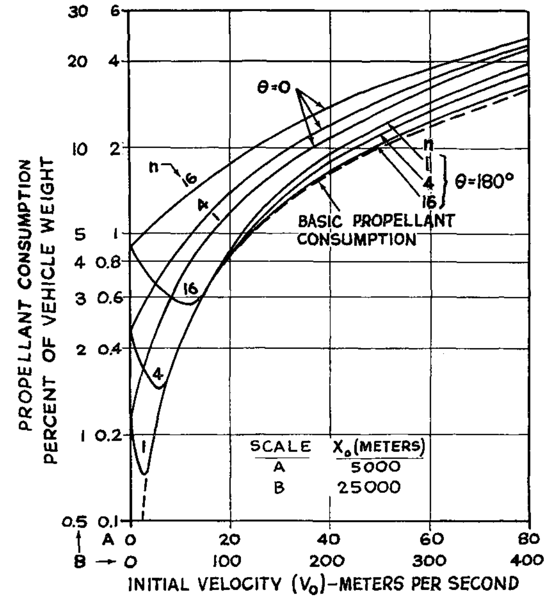


Fig. 6 Propellant consumption, in percent of vehicle weight, as a function of initial velocity for various values of initial position, initial direction of velocity vector, and relative thrust level

Two scales are shown for propellant consumption and initial velocity in Fig. 6. These scales are in direct proportion to the initial distance x_0.

In the case when the vehicle is coasting in the general direction of the satellite and is relatively far away from the satellite, a saving of propellant is possible if initiation of terminal guidance is delayed until the vehicle gets closer to the satellite. The propellant consumption as a function of the angle θ at initiation of terminal guidance is determined as follows.

If the known radial distance and direction of motion of the drifting vehicle at some relatively great distance are x_c and θ_c, respectively, the following equations are obtained from geometry:

$$x_0 \sin\theta = x_c \sin\theta_c \quad (20)$$

$$k \sin\theta = (\alpha x_0/V_0)\sin\theta = (\alpha x_c/V_0)\sin\theta_c = \text{const} \quad (21)$$

where x_0 and θ are values at initiation of terminal guidance. The velocity V_0 during the drift period remains constant (neglecting tidal acceleration).

Curves of the propellant consumption parameter as a function of the angle θ for various values of the constant $k \sin\theta$, or $\alpha x_c \sin\theta_c/V_0$, are shown in Fig. 7. These curves show that propellant consumption is a minimum when the direction of motion θ is between 100° and 150°.

Effect of Tidal Acceleration

1 Guidance coefficient limitations

As the guidance coefficients are decreased, the thrust level likewise will decrease until the tidal acceleration becomes of major importance. A point will be reached where the tidal acceleration predominates so that the vehicle can drift away. Therefore, the tidal acceleration sets a lower limit to the guidance coefficients. The guidance coefficients must be somewhat greater than the theoretically lowest allowable limit in order to prevent excessive wandering and excessive interception time and to allow for instrument inaccuracies.

The low limits of the guidance coefficients will be determined by examination of the one-dimensional equation of motion in the most unfavorable direction. Then a relationship between the two guidance coefficients, or essentially the damping factor, will be established in such a way that they are satisfactory in the most favorable direction. The consideration of one-dimensional motion is sufficient, since the perturbation conditions in other directions are less severe.

The most unfavorable direction is along the radial line from the center of the earth to the satellite, or the η axis. Here the tidal acceleration is repulsive and a maximum (see Fig. 8).

When the tidal acceleration is added to the basic equations of motion, one gets

$$\ddot{x} + a\dot{x} + (b - p)x = 0 \qquad \eta \text{ direction (most unfavorable)} \qquad (22)$$

$$\ddot{x} + a\dot{x} + [b + (p/2)]x = 0 \qquad \xi \text{ direction (most favorable)} \qquad (23)$$

where $p = 2.38 \times 10^{-6}$ sec^{-2} for a 96-min orbit.

Equation (22) shows that the ultimate low limit of b is p. If b is less than p, the vehicle will drift away in general. If b is only slightly greater than p, the vehicle will approach the satellite very slowly in the η direction but much more rapidly in the ξ direction.

In Eqs. (22) and (23), the lowest damping factor occurs in the ξ direction. Since the damping factors are to be unity or greater, the damping factor in the ξ direction will be fixed at unity. Now, the value of the damping coefficient is obtained in terms of b and p as follows:

$$a = 2\alpha = 2\sigma(b + p/2)^{1/2} = 2(b + p/2)^{1/2}$$

Having established the damping coefficient a or α, the various factors for motion in the most unfavorable direction (η direction) become

$$\begin{aligned} \alpha &= (b + p/2)^{1/2} = [(n + \tfrac{1}{2})p]^{1/2} \\ \beta &= (3p/2)^{1/2} \\ \sigma_\eta &= [(b + p/2)/(b - p)]^{1/2} > 1 \end{aligned} \qquad (24)$$

The solution of Eq. (22) is obtained directly from Eqs. (4) by replacing b by $b - p$.

2 *Motion along earth's radius vector*

Since tidal acceleration is largest in earth's radial direction, the propellant consumption for one-dimensional motion in that direction will be determined. The propellant consumption in other directions may be either less or greater, depending on initial conditions. Rotation of ξ, η axes, though important at very low values of guidance coefficients, is neglected.

The equation of motion is given by Eq. (22). The solution of this equation and its first and second time derivatives are

$$x = x_0 e^{-\alpha t}\left[\cosh\beta t + \frac{\alpha}{\beta}\left(1 + \frac{u_0}{\alpha x_0}\right)\sinh\beta t\right]$$

$$\dot{x} = \alpha x_0 e^{-\alpha t}\left[\frac{u_0}{\alpha x_0}\cosh\beta t - \frac{[1 + (u_0/\alpha x_0)]\alpha^2 - \beta^2}{\alpha\beta}\sinh\beta t\right] \qquad (25)$$

$$\ddot{x} = \alpha^2 x_0 e^{-\alpha t}\left[\left(\frac{\alpha^2 - \beta^2}{\beta^2} - \frac{2u_0}{\alpha x_0}\right)\cosh\beta t - \frac{(\alpha^2 - \beta^2) - (u_0/\alpha x_0)(\alpha^2 + \beta^2)}{\alpha\beta}\sinh\beta t\right]$$

where α, β, and σ_η are given by Eqs. (24).

The equation for propellant consumption is

$$\frac{gI_{sp}W_p}{W} = \frac{g}{W}\int_0^\infty |F|\,dt = \int_0^\infty |a\dot{x} + bx|\,dt =$$
$$\frac{x_0(p)^{1/2}}{(n + \frac{1}{2})^{1/2}}\int_0^\infty e^{-\alpha t}\left|\left[n + (2n + 1)\frac{u_0}{\alpha x_0}\right]\cosh\beta t - \frac{\alpha}{\beta}\left[n - 2 + (n + 1)\frac{u_0}{\alpha x_0}\right]\sinh\beta t\right| d(\alpha t) \qquad (26)$$

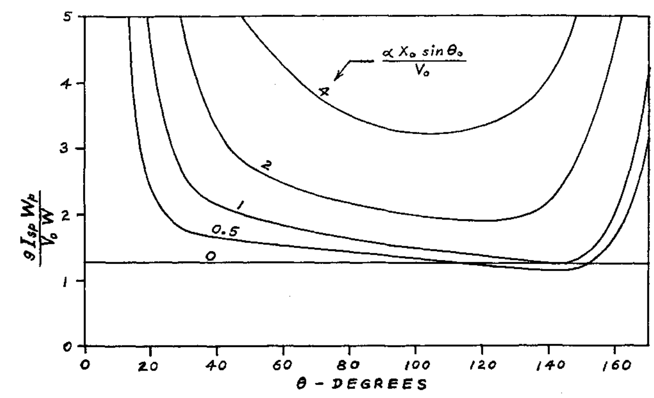


Fig. 7 Effect of ignition delay on propellant consumption; ignition is delayed until direction of velocity vector is θ; the parameter on the curves is a function of conditions at injection phase cutoff; tidal acceleration is neglected

where $n = b/p$ is the relative thrust level, and $u_0 = \dot{x}_0$. Integration of Eq. (26) yields the solid curve for the relative thrust levels $n = 2$ and $n = 4$ shown in Fig. 9.

Now consider the free space rendezvous case, where the guidance coefficients are identical to those shown in Eq. (26). Referring now to Eqs. (2), one obtains

$$b = np \qquad \alpha = [(n + \tfrac{1}{2})p]^{1/2}$$
$$\beta = (\alpha^2 - b)^{1/2} = (p/2)^{1/2}$$

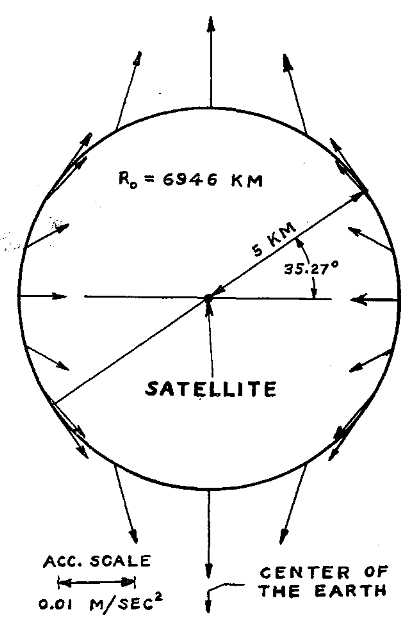
The integral for propellant consumption becomes

$$\frac{gI_{sp}W_p}{W} = \int_0^\infty |a\dot{x} + bx|dt =$$
$$\frac{x_0(p)^{1/2}}{(n + \frac{1}{2})^{1/2}}\int_0^\infty e^{-\alpha t}\left|\left[n + (2n + 1)\frac{u_0}{\alpha x_0}\right]\cosh\beta t - (2n + 1)^{1/2}\left[n + (n + 1)\frac{u_0}{\alpha x_0}\right]\sinh\beta t\right| d(\alpha t) \qquad (27)$$

Integration of Eq. (27) yields the propellant consumption curves shown by broken lines in Fig. 9.

The effect of tidal acceleration on propellant consumption for motion in the earth's radial direction is shown by comparing the solid and broken line curves in Fig. 9 for the two values of relative thrust level, $n = 2$ and 4. In general, the tidal acceleration begins to increase the propellant consumption as the relative thrust level is decreased below 4.

Fig. 8 Direction and magnitude of tidal acceleration at various positions on a circle of 5000-m radius about the satellite; the circle is in any plane containing the earth's radial line



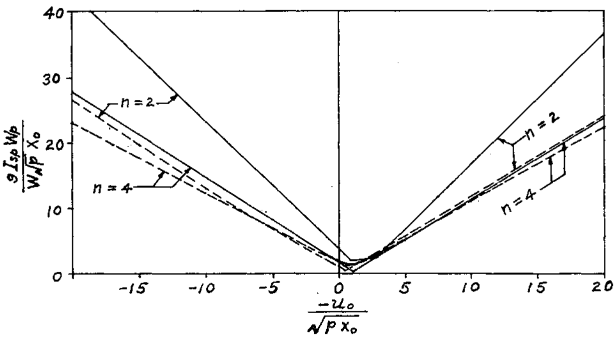


Fig. 9 Effect of tidal acceleration on propellant consumption for one-dimensional exponential motion in earth's radial direction: $n = b/p$ = relative thrust level; ——— = actual propellant consumption; - - - - = free space propellant consumption with guidance coefficients corresponding to n = 2 and 4

3 Delayed ignition—motion parallel to tangent of orbit

The effect of tidal acceleration on the propellant consumption in the case when terminal phase ignition occurs after the vehicle coasts toward the satellite in a line parallel to the trajectory of the satellite now will be considered. Ignition is assumed to occur when the angle θ is 120°, which is not too far from the position for minimum propellant consumption.

If one chooses the x-y coordinate system such that the x axis is parallel to the satellite trajectory (neglecting earth's rotation), the equations of motion and their solutions (with condition that the lowest damping factor is unity) are

$$\begin{aligned} \ddot{x} + a\dot{x} + [b + (p/2)]x &= 0 \\ \ddot{y} + a\dot{y} + (b - p)y &= 0 \end{aligned} \tag{28}$$

$$\begin{aligned} x &= x_0 e^{-\alpha t}\{1 + [1 + (u_0/\alpha x_0)]\alpha t\} \\ y &= y_0 e^{-\alpha t}[\cosh\beta t + (\alpha/\beta)\sinh\beta t] \end{aligned} \tag{29}$$

where

$$\alpha = [(n + \tfrac{1}{2})p]^{1/2} \qquad \beta = (\tfrac{3}{2}p)^{1/2}$$

For $\theta = 120°$ ($y_0 = 3^{1/2} x_0$), the propellant consumption becomes

$$\frac{gI_{sp}W_p}{x_0 p^{1/2} W} = \frac{1}{(n + \frac{1}{2})^{1/2}} \int_0^\infty \left\{\left[n - (2n+1)\frac{u_0}{\alpha x_0} - (n+1)\left(1 - \frac{u_0}{\alpha x_0}\right)\alpha t\right]^2 + 3\left[n\cosh\beta t - \frac{\alpha}{\beta}(n-2)\sinh\beta t\right]^2\right\}^{1/2} e^{-\alpha t}\, d(\alpha t) \tag{30}$$

Curves of this integral for n = 2 and 4 are shown by solid lines in Fig. 10.

The corresponding free space propellant consumption with the same guidance coefficients, α and b, becomes

$$\frac{gI_{sp}W_p}{x_0 p^{1/2} W} = \frac{1}{(n + \frac{1}{2})^{1/2}} \int_0^\infty \left(\left\{\left[n - (2n+1)\frac{u_0}{\alpha x_0}\right] \times \cosh\beta_1 t - (2n+1)^{1/2}\left[n - (n+1)\frac{u_0}{\alpha x_0}\right]\sinh\beta_1 t\right\}^2 + 3n^2\{\cosh\beta_1 t - (2n+1)^{1/2}\sinh\beta_1 t\}^2\right)^{1/2} \cdot e^{-\alpha t} d(\alpha t) \tag{31}$$

where

$$\beta_1 = (p/2)^{1/2} \qquad \alpha/\beta_1 = (2n+1)^{1/2}$$

Curves of this integral are shown in broken lines in Fig. 10.

The effect of tidal acceleration on propellant consumption for a vehicle approaching parallel to the tangent of the trajectory is shown by comparing the solid and broken lines in Fig. 10 for n = 2 and 4. The tidal acceleration again increases the propellant consumption appreciably as the relative thrust level is decreased below 4.

Conclusions

Successful terminal phase rendezvous maneuvers are possible with an exponential type of guidance system, provided that the relative thrust acceleration level is somewhat greater than the tidal acceleration level. Tidal acceleration does not increase the propellant consumption appreciably if the relative thrust level is above 4.

In the case when the initial conditions are 5000 m and 50 m/sec, the propellant consumption is approximately 3% of the vehicle weight. The basic propellant consumption is 2% of the vehicle weight (for a specific impulse of 255 sec).

The basic propellant consumption is determined by the magnitude of the vehicle velocity relative to the satellite. Additional propellant is required whenever a change in the direction of motion is required or an increase in velocity is demanded. This additional propellant consumption decreases as the characteristic time $1/\alpha$ increases, or as the thrust level decreases. As the thrust level approaches the tidal acceleration level, however, the propellant consumption increases again. The lower limit of the thrust level is determined by the tidal acceleration level.

Curves of fuel consumption in parametric form are shown in Figs. 5 and 6. In these curves, tidal acceleration is neglected. These curves are of interest only when the vehicle is within a reasonable distance from the satellite. At great distances, it is preferable to have the vehicle coast, if approaching, until the direction of the velocity vector θ is in the order of 120° (see Fig. 7).

Figures 9 and 10 show that propellant consumption is affected appreciably (increased) by the tidal acceleration when the relative thrust level is well below 4. On the other hand, Fig. 5 shows that if the relative thrust level is much above 4 (large k) propellant consumption is again increased. In the latter case, propellant is wasted in trying to rush the vehicle to the satellite.

Appendix: Tidal Acceleration

The tidal acceleration is the differential gravitational acceleration of two neighboring points in the earth's gravity field. In the present case, it is the acceleration, relative to the satellite, of a free neighboring body. The equations for the tidal acceleration, in terms of space direction fixed coordinates with origin at the satellite, are obtained easily from geometry

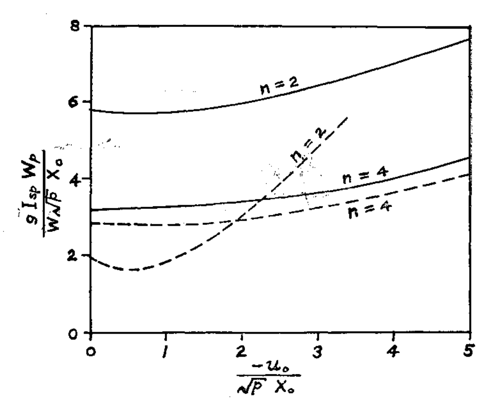


Fig. 10 Effect of tidal acceleration on propellant consumption when ignition is delayed until angle θ is 120°: ——— = actual propellant consumption, and - - - - = free space propellant consumption with guidance coefficients corresponding to n = 2 and 4

(see Fig. 1). The equations are

$$a_\xi = -(p/2)\xi[1 - (3\eta/R_0) + (3/2R_0^2)(4\eta^2 - \xi^2) + \ldots] \cong -(p/2)\xi \quad \text{m/sec}^2$$

$$a_\eta = p\eta[1 + (3/4R_0\eta)(\xi^2 - 2\eta^2) + \ldots] \cong p\eta \quad \text{m/sec}^2$$

where $p = 2gR_e^2/R_0^3 = 2.38 \times 10^{-6}$ sec^{-2} for a 96-min orbit. These are the tidal accelerations in the ξ and η directions, respectively.

It is noted that, at ξ or η = 100 km, the second-order term in the equations affect the tidal acceleration by less than 5%. Therefore, the tidal accelerations are close linear functions of ξ and η.

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Planar Librations of an Extensible Dumbbell Satellite

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If a gravitationally stabilized satellite has a "tip mass" connected to it through a long elastic arm, the librational motion about the local vertical will excite longitudinal or "pumping" vibrations of the elastic arm. If the "pumping" motions are damped out by some type of internal (passive) friction, the librational motion will be damped also. In this paper, the equations of planar motion for such a system are derived and then linearized for the case of viscous damping in a circular orbit. The complete solution is given in terms of two nondimensional parameters, which are a measure of spring stiffness and viscous damping, respectively. It is shown that, if the internal friction arises from "material damping" within the spring, there will be relatively little damping of a viscous nature; however, there is a nonlinear time-independent type of hysteretic damping which could be significant. It is shown how the latter type of damping may be analyzed by a technique of "equivalent viscous damping." A configuration of practical interest is examined in a numerical example.

Nomenclature

A	= amplitude of spring oscillation
a	= parameter defined by Eq. (3.15)
C	= viscous damping constant
C_{cr}	= $2\bar{m}p_\theta(n^2 - 1)^{1/2}$
D	= helical spring coil diameter
D_m	= unit damping energy of material
d	= wire diameter of helical spring
e	= base of natural logarithms
F_d	= damping force
G	= modulus of elasticity in shear (modulus of rigidity)
g_0	= acceleration of gravity at surface of earth
K	= material damping constant
k_s	= spring constant, force/unit extension
m_1	= satellite mass
m_2	= tip mass
$\bar{m}$	= $m_1m_2/(m_1 + m_2)$
n	= $p_s/p_\theta = (k_s/3\bar{m}\Omega^2)^{1/2}$
N_c	= number of coils in helical spring
p_θ	= $3^{1/2}\Omega$ = uncoupled librational circular frequency
p_s	≡ $(k_s/\bar{m})^{1/2}$ = uncoupled pumping circular frequency
q_r, q_θ	= generalized coordinates
Q_r, Q_θ	= generalized forces
r	= distance between masses m_1 and m_2
r_0	= value of r when spring is unstrained
R_0	= radius of earth
S	= spring force
s, s_{ij} $(i = 1,2)$	= roots of characteristic equation
T	= kinetic energy of system
T_c	= kinetic energy of mass center
T_1, T_2	= kinetic energies of m_1 and m_2 with respect to mass center
t	= time
T_0	= orbital period
U	= potential energy stored in spring
ΔU	= energy loss/cycle
x	= extension of spring from free length
x_{st}	= $r_0/(n^2 - 1)$ = static extension of spring in orbit
z	= $(x - x_{st})/r_0$
Z	= initial value of z
$\alpha_i(i = 1,2)$	= negative of real part of root of characteristic equation
$\beta_i(i = 1,2)$	= imaginary part of root of characteristic equation
γ	= $C/\bar{m}p_\theta$
$\delta(\ )$	= virtual displacement in ( )
ζ	= $\|\mu_1\|n^2/\pi\beta_1(n^2 - 1)^{1/2}$
Θ	= initial libration amplitude
θ	= libration angle
θ_1, θ_2	= constants of integration
θ_m	= $\Theta e^{-t/\tau}$
Λ	= r_0d/N_cD^2
λ	= $\frac{1}{2}\Delta U/U$ = logarithmic decrement
λ_m	= logarithmic decrement of material
μ	= Z/Θ
μ_1, μ_2	= μ evaluated for s_{11} or s_{21}
ν	= $g_0R_0^2$
ρ, ρ_1, ρ_2	= radius from center of earth to mass center of satellite, m_1, m_2, respectively

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